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EQUIVALENCE COLOURING OF GRAPHS

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ABSTRACT

Let G = (V, E) be a simple and undirected graph. A proper colouring of the vertices of V (G) is an assignment of colours to the vertices of G such that adjacent vertices receive different colours. A proper colouring of G induces a partition of V (G) into independent sets. The minimum cardinality of a proper colour partition of G is called the chromatic number of G and is denoted by $\chi(G)$. If in a proper colour partition of G, the union of any two-colour classes induces an acyclic subgraph, then the colouring is called acyclic colouring of G. {[4], [5], [6]}. If instead, the union of any two colour classes in a proper colour partition induces a disjoint collection of stars, the resulting proper colour partition is called a star partition. {[6]}. A subset S of V(G) is called an equivalence set if the subgraph induced by S is component wise complete. In this paper, a study of proper colour partition in which the union of any two colour classes induces an equivalence set is initiated.

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INTRODUCTION

In what follows, a graph G means a finite, simple and undirected graph. The chromatic number of G is the minimum cardinality of a partition of V (G) into independent sets. If the subgraph induced by a set of vertices of G is component wise complete, then that set is called an equivalence set. In a proper colour partition, the subgraph induced by the union of any two-colour classes is an equivalence set, then that colouring is called an equivalence colouring of the graph. In any graph, the partition of V (G) into subsets each of which is a singleton is obviously an equivalence colouring. The minimum cardinality of an equivalence colour partition of G is called the equivalence chromatic number of G and is denoted by χ_{eq}(G).

A study of this colouring is made in this paper.

Definition 1.1. A proper colouring is an equivalence colouring if the union of any twocolour classes induce an equivalence set. The minimum cardinality of such a colouring is called equivalence chromatic number of a graph and is denoted by χ_{eq}(G)

Example 1.1. Consider P_6 with vertices $\{v_1, v_2, v_3, v_4, v_5, v_6\}$. $\chi(P_6)$ = 2 and the two colour classes are $\{v_1, v_3, v_5\}$ and $\{v_2, v_4, v_6\}$. The union of these twocolourclasses induce P_6 which is not an equivalence set. But $\{v_1, v_4\}$, $\{v_2, v_5\}$, $\{v_3, v_6\}$ is a χ_{eq}– partition of P_6.

Remark 1.2. $\chi(G) \le \chi_{eq}(G)$, When $(G) = K_n$, $\chi(G) = \chi_{eq}(G)$.

Remark 1.3. This study is similar to acyclic and star colouring of graphs [8th Cologne-

Twente Workshop on Graphs and Combinatorial Optimization CTW09, Ecole Poly technique and CNAM, Paris,

France, June 2 − 4, 2009, A. Lyons, Acyclic and star colouring of Joins of graphs and algorithm K for cographs, 199 − 204.]

In a minimum equivalence color partition, the sub graph induced by the union of any two colour classes is of form $tK_2 \cup sK_1$, t, s ≥ 0. In a minimum equivalence color partition, there may exist two independent colour classes without any edges between them. In such a case, there will be an induced P_3 in the graph. For example, let G be the graph obtained from C_4 by attaching paths of length 2 one each at the diametrically opposite vertices of C_4.

Let V $(C_4) = \{u_1, u_2, u_3, u_4\}$. Let $\{u_2, u_5, u_6\}$ and $\{u_4, u_7, u_8\}$ be the path of length of 2 attached at u_2 and u_4 respectively. $\{\{u_1, u_6, u_8\}, \{u_2, u_7\}, \{u_4, u_5\}, \{u_3\}\}$;

$\{\{u_1, u_6\}, \{u_2, u_7\}, \{u_3, u_8\}, \{u_4, u_5\}\}$ are two minimum equivalence colour partition. In the first one, there is no edge between $\{u_1, u_6, u_8\}$, and $\{u_3\}$. In the second one, there is no edge between $\{u_1, u_6\}$ and $\{u_3, u_8\}$. In this graph, $\prec \{u_1, u_2, u_3, u_4\} \succ$ is an induced P_3. The join of $\{u_1, u_6, u_8\}$ and $\{u_3\}$ is independent but u_2 is adjacent with both u_1 and u_3 of the join and hence the number of classes cannot be reduced. Similarly, in the join of $\{u_1, u_6\}$ and $\{u_3, u_8\}$ which is independent, but u_2 is adjacent with both u_1 and u_3 of the join and hence the number of classes cannot be reduced.

Theorem 1.4. If a graph G is induced P_3 − free, then in a minimum equivalence colour partition, (i) There exists an edge between any twocolour classes. (ii) Every colour class contains a colourful vertex, that is, a vertex which is adjacent with every other colour class. (iii) After suitable modification of the minimum equivalence colour partition, there exists a colour class which is an equivalence dominating set of the graph.

Proof

i. Suppose π = $\{V_1, V_2, \dots, V_k\}$ be a minimum equivalence colour partition in an induced P_3-free graph G. Suppose there exists no edge between V_i and V_j, {1 ≤ i, j ≤ k, i≠ j}. Consider $\pi_1 = \pi \cup V_i \cup V_j - \{V_i - V_j\}$. As G is induced P_3- free, there does not exist a vertex u in any V_r {1 ≤ r ≤ k, r ≠ i, j} such that u is adjacent with some vertex of V_i and some vertex V_j. Therefore, π_1 is an equivalence colour partition with cardinality less than that of the minimum equivalence colour partition π, a contradiction.

ii. Arguing as in (i), we get that every colour class contains a colourful vertex (since as G is induced P_3- free, any vertex u in V_i which is not adjacent to V_j can be included in V_j without aecting equivalence nature of π).

iii. Arguing as in (i), any vertex not in V_1 which is not adjacent with V_1 can be moved to V_1 resulting in V_1, an equivalence dominating set of the graph.

Remark 1.5. (i)leads to achromatic equivalence colour partition in an induced P_3 graph.

(ii)leads to b-equivalence colouring partition in an induced P_3 graph.

Also, Greedy equivalence colouring partition in an induced P_3 graph can be defined.

χ_{eq}(G) for some Well-known Graphs

Observation 1.6.

- $\chi_{eq}(K_n)$= n.
- $\chi_{eq}(K_{1,n})$ = n + 1.
- $\chi_{eq}(K_{m,n})$ = m + n.
- $\chi_{eq}(W_n)$= n for all n ≥ 4.
- $\chi_{eq}(P_n) = \begin{cases} 2 \; if n = 2 \\ 3 \; if n \geq 3 \end{cases}$
- (vi)$\chi_{eq}(C_n) = \begin{cases} 3 \text{ if n } \equiv 0 \text{ (mod3)} \\ 4 \text{ if n } \equiv 1 \text{ (mod3)}) \\ 5 \; if \; n \equiv 2 \; (mod3) \end{cases}$
- $\chi_{eq}(G \circ K_1) = \chi_{eq}(G)$+ 1. In particular, $\chi_{eq}(G \circ K_1)$ = m + 1.

Definition 1.7. An independent subset S of V (G) is an equivalence independent set of G if |N(v) ∩ S| ≤ 1 for every (v) in (V − S). That is, any equivalence independent subset of V (G) is a nearly perfect set of (G). An equivalence independent subset of G is an independent semi-strong subset of (G). The maximum cardinality of an equivalence independent set is called the equivalence independence number of (G) and is denoted by β_{eq}(G) (or also by iss(G)). A maximal equivalence independent set of G need not be a dominating set of G.

Example 1.2. In C_4, any single vertex constitutes a maximum equivalence independent set of C_4. It is obviously not dominating and the diametrically opposite vertex is independent of the singleton equivalence independent set and its inclusion will result in a vertex in the complement having two neighbours in the two-element independent set.

Remark 1.8. By recalling the definition of perfect dominating set : A dominating subset D of nearly perfect if for any v in V − D, |N(v)∩D| ≤ 1, a subset D is strongly stable if for any v in V (G), |N(v) ∩ D| ≤ 1. [Page 115, 116 of Chapter 4.2 of Fundamentals of domination in graphs] The values of β_{eq}(G) are found for some known classes of graphs which are similar to the values of iss(G) [7].

β_{eq}(G) For some Known Classes of Graphs

- (1) $\beta_{eq}(K_n)$ = 1.
- (2)$\beta_{eq}(K_{m,n})$ = 1.
- (3)$\beta_{eq}(K_{1,n})$ = 1.
- (4) $\beta_{eq}(P_n) = \begin{cases} \frac{n}{2} - 1 \; if \; n \equiv 2(mod\;4), n \neq 6 \\ \left\lceil \frac{n}{2} \right\rceil \; otherwise \end{cases}$
- (5) $\beta_{eq}(W_n)$ = 1 for all n ≥ 4.

- (6) $\beta_{eq}(C_n) = \left\lfloor\frac{n}{3}\right\rfloor$
- (7) β_{eq}(P) = 1, where P is the Petersen Graph.
- (8) $\beta_{eq}(K_m(a_1, a_2, \ldots\ldots a_m)) = m$
- (9) $\beta_{eq}(K_{a_1, a_2, \ldots\ldots a_m})$= 1, if n ≥ 3.
- (10) β_{eq}([illegible] $\circ K_1$) = n. In particular $\beta_{eq}(K_m \circ K_1) = m$.
- (11) β_{eq}([illegible]) = n.

Main Results

Theorem 1.9: In any graph G, $(n/\beta_{eq}(G)) \leq \chi_{eq}(G) \leq (n - \chi_{eq}(G) + 1)$ Proof. Let Π = $\{V_1, V_2, V_3, \ldots\ldots V_k\}$be a χ_{eq}- partition of G where k = χ_{eq}(G). Then $|V_i| \leq \beta_{eq}$(G) for every i, (1 ≤ i ≤ k). Therefore, $n = \sum_{i=1}^{k} |V_i| \leq k\, \beta_{eq}(G)$.

Hence$(n/\beta_{eq}(G)) \leq k = \chi_{eq}(G)$. Consider the partition $\prod_1 = \{V_1, V_2, V_3, \ldots\ldots V_r\}$ where V_1 is aβ_{eq}- set of G and the remaining are singletons from V − V_1.$\prod_1$ is an equivalence colouring and hence $\chi_{eq}(G) \leq r = (n - \chi_{eq}(G) + 1)$.

Remark 1.10. χ_{eq}(G)= n if and only ifβ_{eq}(G) = 1.

Theorem 1.11. β_{eq}(G) = 1 if and only if G = K_n or for any independent set S of G with cardinality ≥ 2, there exists a vertex in V − S which is adjacent with at least two vertices of S.

Proof. If G = K_n or for any independent set S of G with cardinality ≥ 2, there exists a vertex in V −S which is adjacent with at least two vertices of S, then β_{eq}(G) = 1. Conversely, suppose

β_{eq}(G) = 1. If G ≠ K_n then, G has an independent set say S of cardinality ≥ 2. If any vertex of V − S is adjacent with at most one vertex of S, then S is equivalence independent set and hence$\beta_{eq}(G) > |S| \geq 2$, a contradiction. Hence the theorem.

Remark 1.12. $\beta_{eq}(G) \leq \beta_0(G)$. In$K_n$, β_{eq}(G) $\beta_0(G)$. In C_4, $\beta_{eq}(G) = 1 \leq \beta_0(G) = 2$.

Theorem 1.13. $\beta_{eq}(G) = 2$ if and only if there exist two independent vertices u and v such that any vertex w in V − S is adjacent with at most one of (u, v) and for any two independent vertices of ⟨V - S⟩, either one of (x, y) is adjacent with one of u, v say u and at least one vertex of (V − S) − (x, y) is adjacent with at least two of v, x and y (or)u, x and y.

Proof. Let $\beta_{eq}(G) = 2$. Then there exists an independent subset S = u, v such that any vertex w in V − S is adjacent with at most one of (u, v).consider ⟨V - S⟩. If V −S is empty, then

G =($\overline{K_2}$). If $|V - S| = 1$, then G =$(\overline{K_3})$ or $(K_2 \cup K_1)$.

Suppose $|V - S| = 2$. Let (V − S) = (x, y).

Case 1: x and y are adjacent.

Subcase 1: u, v are independent of x, y. Then $\beta_{eq}(G) > 2$, a contradiction.

In other cases, we get either P_3 with an isolated vertex or a C_3 with an isolated vertex or a P_4.

In all these cases $\beta_{eq}(G) = 2$

Case 2: x and y are independent.

In this case, either G is $2K_2$ or a P_3 with an isolated vertex. In this case, $\beta_{eq}(G) = 2$.

Suppose |V − S| = 3.

Subcase 1: Suppose □V - S□ is complete. Then $\beta_{eq}(G) = 2$ except when u and v are isolates.

Subcase 2: Suppose □V - S□ is not complete. The following 13 graphs satisfy $\beta_{eq}(G) = 2$.

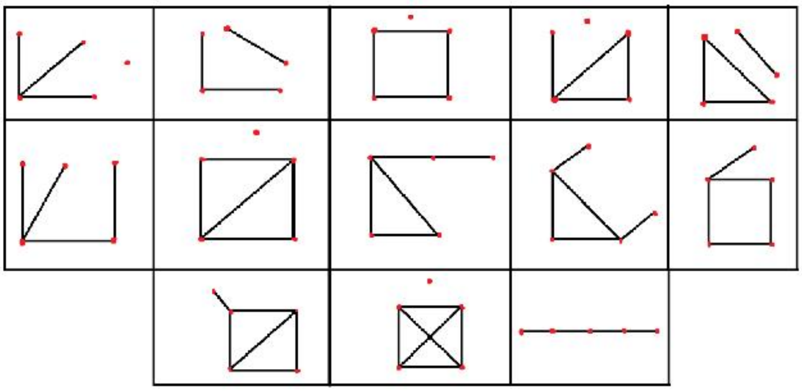

Figure 1

Let |V − S| = 4. Suppose □V - S□ is complete. Then no vertex of (V − S) is adjacent with both u and v. Also, u and v cannot be both independent of (V − S).

Suppose □V - S□ is not complete. Then there exist x and y in (V − S) which are independent vertices in □V - S□.If u is adjacent with x then at least one vertex of (V − S) − (x, y) is adjacent with at least two of v, x and y. If both u and v are not adjacent with any of x, y, then at least one vertex of (V − S) − (x, y) is adjacent with atleast two of v, x and y or u, x and y.

Thus, for any two independent vertices of □V - S□, either one of (x, y) is adjacent with one of u, v say u and at least one vertex of (V − S) − (x, y) is adjacent with at least two of v, x and y or u, x and y. The converse is obvious.

Example 1.3. Let G be obtained from K_4 by adding two pendent vertices one each at two of the vertices of K_4. Then $\beta_{eq}(G) = 2$.

Example 1.4. Let $V_{K_{1,3}} = (u, v_1, v_2, v_3)$. Let G be obtained from $K_{1,3}$by adding two pendent vertices x, y one each at v_1 and v_2. Then $\beta_{eq}(G) = 2$.

Theorem 1.14. $\beta_{eq}(G) = n$ if and only if G = $(\overline{K_n})$.

Proof. Suppose, $\beta_{eq}(G) = n$. As $\beta_{eq}(G) \leq \beta_0(G)$, we get that $\beta_0(G)$ = n.Therefore, G = $(\overline{K_n})$ The converse is obvious.

Theorem 1.15. $\beta_{eq}(G)$ = n − 1 if and only if G has exactly one edge.

Proof. Suppose $\beta_{eq}(G) = n - 1$. Then, as $\beta_{eq}(G) \leq \beta_0(G)$, we get that $\beta_0(G) = n - 1$ or n.

If $\beta_0(G) = n$, then G = $(\overline{K}_n)$ and hence $\beta_{eq}(G) = n$, a contradiction. Therefore,

$\beta_0(G) = n - 1$. That is, G has exactly one edge. The converse is obvious.

The following two theorems are easy to prove.

Theorem 1.16. Let G_1 amd G_2 be two vertex disjoint graphs. Then

$\chi_{eq}(G_1 \cup G_2) = max\,(\chi_{eq}(G_1), \chi_{eq}(G_2))$.

Theorem 1.17. Let G_1 amd G_2 be two vertex disjoint graphs. Then

$\chi_{eq}(G_1 + G_2) = |V(G_1)| + |V(G_2)|$

Topic for further study

- Maximum cardinality of equivalence colour partition in which there exists an edge between any two colour classes in P_3 −free graph.
- Grundy equivalence colour partition.

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